

THE DISTANCE RATIO METRIC ON SOME DOMAINS OF \mathbb{C} PLANE

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Abstract We prove a generalization for some \mathbb{C} domains of Gehring - Palka theorem on Möbius transformations regarding the distance ratio metric. Namely, we show that this theorem is valid for arbitrary holomorphic mappings $f : \mathbb{H} \rightarrow \mathbb{H}$ or $f : \mathbb{D} \rightarrow \mathbb{D}$ with the same Lipschitz constant 2.

1. Introduction

For a subdomain $G \subset \mathbb{R}^n$ and for all $x, y \in G$ the distance ratio metric j_G is defined as

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right),$$

where $d(x, \partial G)$ denotes the Euclidean distance from x to ∂G . The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [3] and in the above simplified form by M. Vuorinen [9]. As the "first approximation" of the quasihyperbolic metric, it is frequently used in the study of hyperbolic type metrics ([1],[2],[6],[10]) and geometric theory of functions.

For an open continuous mapping $f : G \rightarrow G'$ we consider the following condition: there exists a constant $C \geq 1$ such that for all $x, y \in G$ we have

$$j_{G'}(f(x), f(y)) \leq C j_G(x, y),$$

or, equivalently, that the mapping

$$f : (G, j_G) \rightarrow (G', j_{G'})$$

between metric spaces is Lipschitz continuous with the Lipschitz constant C .

However, unlike other frequently used metrics (hyperbolic metric, absolute ratio metric), the distance ratio metric j_G is not invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for this metric under conformal mappings or Möbius transformations in higher dimension. F.W. Gehring, B.P. Palka and B.G. Osgood proved that j metric is not changed by more than a factor 2 under Möbius transformations, see [2], [3]:

Theorem A *If G and G' are proper subdomains of \mathbb{R}^n and if f is a Möbius transformation of G onto G' , then for all $x, y \in G$*

$$j_{G'}(f(x), f(y)) \leq 2 j_G(x, y)$$

It is an interesting problem to investigate Lipschitz continuity of the distance-ratio metric under some other conformal mappings.

In this paper we show that analogous to the above results are valid for *arbitrary holomorphic self-mappings* of a half-plane or the open unit disk. Thereby Theorem A is considerably generalized in these cases.

2. Results and proofs

We shall treat firstly the case of upper half-plane $\mathbb{H} := \{z | \Im z > 0\}$.

A generalization of Theorem A for this domain is given by the following

Theorem 1 *For any holomorphic mapping $f, f : \mathbb{H} \rightarrow \mathbb{H}$, and all $z, w \in \mathbb{H}$, we have*

$$j_{\mathbb{H}}(f(z), f(w)) \leq 2j_{\mathbb{H}}(z, w). \quad (1.1)$$

The constant 2 is best possible.

Proof

Denote $s := \min\{\Im z, \Im w\}$, $S := \min\{\Im f(z), \Im f(w)\}$, $T := \max\{\Im f(z), \Im f(w)\}$.

Then

$$j_{\mathbb{H}}(z, w) = \log\left(1 + \frac{|z - w|}{s}\right); \quad j_{\mathbb{H}}(f(z), f(w)) = \log\left(1 + \frac{|f(z) - f(w)|}{S}\right).$$

Main tool in the proof will be Julia's variant of the famous Schwarz-Pick Lemma for the half-plane [cf 2].

Theorem B *For any holomorphic mapping $f, f : \mathbb{H} \rightarrow \mathbb{H}$, and all $z, w \in \mathbb{H}$, we have*

$$\left| \frac{f(z) - f(w)}{f(z) - \overline{f(w)}} \right| \leq \left| \frac{z - w}{z - \overline{w}} \right|.$$

Applying this assertion, we get

$$\left| \frac{f(z) - \overline{f(w)}}{f(z) - f(w)} \right|^2 - 1 \geq \left| \frac{z - \overline{w}}{z - w} \right|^2 - 1,$$

that is

$$\left| \frac{f(z) - f(w)}{z - w} \right|^2 \leq \frac{\Im f(z) \Im f(w)}{\Im z \Im w},$$

since for any $x, y \in \mathbb{C}$, the identity

$$|x - \overline{y}|^2 - |x - y|^2 = 4\Im x \Im y$$

holds.

Therefore,

$$\frac{|f(z) - f(w)|^2}{S^2} \leq \frac{|z - w|^2}{s^2} \frac{T}{S} = \frac{|z - w|^2}{s^2} \left(1 + \frac{T - S}{S}\right),$$

i.e.,

$$\frac{|f(z) - f(w)|}{S} \leq \frac{|z - w|}{s} \sqrt{1 + \frac{|f(z) - f(w)|}{S}}, \quad (1.2)$$

since $T - S = |\Im(f(z) - f(w))| \leq |f(z) - f(w)|$.

Denoting $\frac{|z-w|}{s} := 2X \in [0, \infty)$, the relation (1.2) gives

$$1 + \frac{|f(z) - f(w)|}{S} \leq (X + \sqrt{1 + X^2})^2.$$

Hence,

$$j_{\mathbb{H}}(f(z), f(w)) = \log\left(1 + \frac{|f(z) - f(w)|}{S}\right) \leq 2\log(X + \sqrt{1 + X^2}) \leq 2\log(1 + 2X) = 2j_{\mathbb{H}}(z, w),$$

and the proof is done.

To prove that the constant 2 is best possible, choose

$$f_0(z) = a - \frac{1}{b + z},$$

where a and b are arbitrary real numbers.

Since $\Im f_0(z) = \frac{\Im z}{|b+z|^2}$, we conclude that f_0 maps the upper half-plane into itself.

A calculation of j values along the line $\zeta \subset \mathbb{H}$ given by $\zeta := \{z = i - b + t, t \in \mathbb{R}\}$ with $w := i - b$, gives

$$j_{\zeta}(z, w) = \log(1 + t); \quad j_{\zeta}(f_0(z), f_0(w)) = \log\left(1 + \frac{\frac{t}{\sqrt{1+t^2}}}{\frac{1}{1+t^2}}\right) = \log(1 + t\sqrt{1+t^2}).$$

Hence,

$$\frac{j_{\zeta}(f_0(z), f_0(w))}{j_{\zeta}(z, w)} = \frac{\log(1 + t\sqrt{1+t^2})}{\log(1 + t)},$$

and this expression is tending to 2 as $t \rightarrow \infty$.

It follows that the constant 2 is best possible.

A generalization of Theorem A for the unit disk case is given by the following

Theorem 2 *For any holomorphic mapping $f, f : \mathbb{D} \rightarrow \mathbb{D}$, and all $z, w \in \mathbb{D}$, we have*

$$j_{\mathbb{D}}(f(z), f(w)) \leq 2j_{\mathbb{D}}(z, w). \quad (2.1)$$

Proof

Let $\max\{|z|, |w|\} = r$ and suppose that $|f(z)| \geq |f(w)|$. Then

$$j_{\mathbb{D}}(z, w) = \log\left(1 + \frac{|z - w|}{1 - r}\right); \quad j_{\mathbb{D}}(f(z), f(w)) = \log\left(1 + \frac{|f(z) - f(w)|}{1 - |f(z)|}\right).$$

The proof is based on well-known Schwarz-Pick lemma, stated in the following form

Theorem C *Let f be a holomorphic mapping of the open unit disk \mathbb{D} into itself. Then for any $z_1, z_2 \in \mathbb{D}$, we have*

$$\left| \frac{f(z_2) - f(z_1)}{1 - \bar{f}(z_1)f(z_2)} \right| \leq \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|.$$

An application of this lemma gives

$$\left| \frac{f(z) - f(w)}{1 - \bar{f}(z)f(w)} \right|^{-2} - 1 \geq \left| \frac{z - w}{1 - \bar{z}w} \right|^{-2} - 1,$$

that is,

$$\begin{aligned} \left| \frac{f(z) - f(w)}{z - w} \right|^2 &\leq \frac{(1 - |f(z)|^2)(1 - |f(w)|^2)}{(1 - |z|^2)(1 - |w|^2)} \\ &\leq \frac{(1 - |f(z)|^2)(1 - |f(w)|^2)}{(1 - r^2)^2}, \end{aligned}$$

where we used the following identity for complex numbers x, y ,

$$|1 - \bar{x}y|^2 - |x - y|^2 = (1 - |x|^2)(1 - |y|^2).$$

Therefore,

$$\frac{|f(z) - f(w)|}{1 - |f(z)|} \leq \frac{|z - w|}{1 - r} \frac{\sqrt{(1 + |f(z)|)(1 + |f(w)|)}}{1 + r} \sqrt{\frac{1 - |f(w)|}{1 - |f(z)|}},$$

i.e.,

$$\frac{|f(z) - f(w)|}{1 - |f(z)|} \leq \frac{|z - w|}{1 - r} \frac{1 + |f(z)|}{1 + r} \sqrt{1 + \frac{|f(z) - f(w)|}{1 - |f(z)|}}, \quad (2.2)$$

since

$$\frac{1 - |f(w)|}{1 - |f(z)|} = 1 + \frac{|f(z)| - |f(w)|}{1 - |f(z)|} \leq 1 + \frac{|f(z) - f(w)|}{1 - |f(z)|}.$$

To obtain an estimation for $|f(z)|$, $z \in \mathbb{D}$, suppose that $f(0) = a \in \mathbb{D}$. Applying Theorem C with $z_1 = 0, z_2 = z$, we get

$$|z| \geq \frac{|f(z) - a|}{|1 - \bar{a}f(z)|} \geq \frac{|f(z)| - |a|}{1 - |a||f(z)|},$$

that is,

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \leq \frac{r + |a|}{1 + |a|r} \quad (2.3)$$

By (2.3) we get $\frac{1+|f(z)|}{1+r} \leq \frac{1+|a|}{1+|a|r} := 2c(a, r) = 2c$ and, denoting $\frac{|z-w|}{1-r} := X \in [0, +\infty)$, the inequality (2.2) gives

$$1 + \frac{|f(z) - f(w)|}{1 - |f(z)|} \leq (cX + \sqrt{1 + c^2 X^2})^2.$$

Hence,

$$\frac{j_{\mathbb{D}}(f(z), f(w))}{j_{\mathbb{D}}(z, w)} \leq \frac{2 \log(cX + \sqrt{1 + c^2 X^2})}{\log(1 + X)}.$$

Now it should be notified that the function g ,

$$g(X) := cX + \sqrt{1 + c^2 X^2} - (1 + X)$$

is negative for $X \in (0, T)$, where $T = \frac{2(1-c)}{2c-1} = \frac{2|a|r+1-|a|}{|a|(1-r)}$.

Since $X = \frac{|z-w|}{1-r} \leq \frac{2r}{1-r}$, it follows that $X \in (0, T)$, i.e., $\log(cX + \sqrt{1 + c^2 X^2}) \leq \log(1 + X)$,

which provides the proof.

Remark We cannot claim in this case that the best possible Lipschitz constant is $C^* = 2$ since we do not know an example of analytic function f_0 , $f_0 : \mathbb{D} \rightarrow \mathbb{D}$ such that

$$\sup_{z, w} \frac{j_{\mathbb{D}}(f_0(z), f_0(w))}{j_{\mathbb{D}}(z, w)} = 2.$$

This problem was recently treated in [11] where the following estimation for $C^* = C^*(a)$ is proved

$$1 + |a| \leq C^* \leq \min\{2(1 + |a|), \sqrt{5 + 2|a| + |a|^2}\}; \quad a := f(0).$$

By Theorem 2 this is reduced to

$$1 + |a| \leq C^* \leq 2,$$

but problem of determining best possible Lipschitz constant for the unit disk remains open.

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